ISSN: 2277-9655
Scientific Journal Impact Factor: 3.449
(ISRA), Impact Factor: 1.852

IJESRT

INTERNATIONAL JOURNAL OF ENGINEERING SCIENCES & RESEARCH TECHNOLOGY

Estimation of Constant-Stress Partially Accelerated Life Test Plans for Rayleigh Distribution using Type-II Censoring

Sadia Anwar *, Arif Ul Islam
Department of Statistics & O.R, Aligarh Muslim University , Aligarh, India
sadiaanwar97@yahoo.com

Abstract

This study deals with simple Constant Stress Partially Accelerated life test (CSPALT) using type-II censoring. The lifetime distribution of the test item is assumed to follow Rayleigh distribution. The Maximum Likelihood (ML) Estimation is used to estimate the distribution parameters and acceleration factor. Asymptotic confidence interval estimates of the model parameters are also evaluated by using Fisher information matrix. In the last a simulation study is performed to illustrate the statistical properties of the parameters.

Keywords: Acceleration factor; Maximum likelihood estimation; Reliability function; constant stress; Fisher Information matrix; Asymptotic Confidence Interval, Simulation Study

Introduction

Manufacturing designs are improving continuously because of the innovation in technology; therefore, it is becoming more and more difficult to obtain information about lifetime of products or materials with high reliability at the time of testing under normal conditions. These conditions are referred to as stresses, which may be in the form of temperature, voltage, force, humidity, pressure, vibrations, etc. In such problems, accelerated life tests (ALTs) are often used to quickly obtain information on the life time distribution of products by testing them at accelerated conditions than normal use conditions to induce early failures**.**

In general, there are three types of stress loading applied in accelerated life test. First is the constant stress accelerated lie test in which the products are operated at fixed stress levels throughout the test.The second is the stepp stress ALT, In this type of ALT the stress on a unit is increased step by step until it fails and the third is progressive stress ALT. There are two types of data are obtained from ALT, first is the complete in which failure time of each sample unit is observed or known and second is the censored data in which failure time of each sample unit may not be available or observed. In ALT , the mathematical model relating to the lifetime of an item and stress is known or can be assumed.But,in some cases these these relationships are not known and can not be assumed, i.e the data obtained from ALT can not be extrapolated to use condition.So,partially accelerated life test can be used in such cases in which the test items are run at both normal and higher than normal stress conditions.There are two commonly used methods in PALT, i.e. constant stress PALT and step stress PALT. In constant stress PALT products are tested at either usual or higher than usual condition only until the test is terminated.The another approach to accelerate failures is the step stress which increases the load applied to the products in a specified discrete sequence. A sample of test items is first run at use condition and, if it does not fail for a specified time, then it is run at accelerated condition until a prespecified numbers of failures are obtained or a prespecified time has reached.

Constant stress PALT has been studied by many authors.for example Bai and Chung [1] discussed the optimall designing constant strss PALT or the test item having exponential distribution under Type I censoring.Bai et.al [2] discussed the PALT plan for lognormal distribution under time censored data.after that Bai et al [3] also considered the problemof failure-censored accelerated life-test sampling plans for lognormal and Weibull distributions. Abdel-Ghani [4] investigated some lifetime models under partially accelerated life tests. Ghaly et al. [5] discussed the PALT problem of parameter estimation for Pareto using Type I censoring and after that

ISSN: 2277-9655
Scientific Journal Impact Factor: 3.449
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Ghaly et al. [6] considered the same problem under Type II censoring. Ismail [7] used the maximum likelihood method to estimate the acceleration factor and parameters of the Pareto distribution under PALT. Ismail [8] discuss the constant stress PALT for the Weibull failure distribution under failure censored case. Ismail (9) considered the problem of optimally designing a simple time-step-stress PALT which terminates after a pre-specified number of failures and developed optimum test plans for products having a two-parameter Gompertz lifetime distribution. Zarrin et al. [10] considered constant stress PALT with type-I censoring. Assuming Rayleigh distribution as the underlying lifetime distribution, the MLEs of the distribution parameter and acceleration factor were obtained. More recent Saxena et al [11] consider the PALT design for extreme value distribution using type I censoring and Kamal et al. [12] discuss the same problem for Inverted Weibull distribution.

This work was conducted for constant-stress PALT under type II censored sample. the problems of estimation in constant stress PALT are considered under Rayleigh distribution. Maximum likelihood estimates and confidence intervals for parameters and acceleration factor are obtained. The performance of the inference method used in present papers is evaluated by a simulation study.

Model description

The Rayleigh Distribution

The lifetimes of the test items are assumed to follow a Rayleigh distribution. The probability density function (pdf) of the Rayleigh distribution is given by

$$f(t)=\frac{t}{\theta^2}\exp\left(-\frac{t^2}{2\theta^2}\right), \qquad 0\le t<\infty, \theta>0 \quad (1)$$

and the cumulative distribution function (cdf) is given by

$$F(t)=1-\exp\left(-\frac{t^2}{2\theta^2}\right), \qquad 0\le t<\infty, \theta>0 \quad (2)$$

The reliability function of the Rayleigh distribution is given by

$$F(t)=1-\exp\left(-\frac{t^2}{2\theta^2}\right), \qquad 0\le t<\infty, \theta>0 \quad (3)$$

and the corresponding hazard rate is given by

$$h(t)=\frac{t}{\theta^2}$$

The Rayleigh distribution is very significant in modeling the lifetime of random phenomena. It arises in many areas of applications, including reliability, life testing and survival analysis. Rayleigh distribution is a special case of Weibull distribution (shape parameter=2).

Constant stress PALT

- **Test procedure**

In the constant-stress PALT, all of the n items are divided into two parts, in which nk items are randomly chosen among n items, which are allocated to accelerated conditions and the remaining $n(1-k)$ are allocated to normal use conditions, where k is proportion of sample units allocated to accelerated condition. Each test item is run until the occurrence of r number of failures and the test condition is not changed.

- **Assumptions**

There are some assumptions also made in a constant-stress PALT.

❖ The lifetimes T_i , $i=1,2,\ldots\ldots\ldots n(1-k)$ of items allocated to normal use condition, are i.i.d. random variables.

❖ .The lifetime X_j , $j=1,2,\ldots\ldots\ldots nk$ of items allocated to accelerated condition, are i.i.d random variables.

❖ The lifetimes T_i and X_j are mutually statistically-independent

Maximum likelihood estimation

The maximum likelihood estimation (MLE) is the most important and widely used method in statistics. The idea behind the maximum likelihood parameter estimation is to determine the estimates of the parameter that maximizes the likelihood of the sample data. Also the MLEs have the desirable properties of being consistent and asymptotically normal for large samples

In a simple constant-stress PALT, the test item is run either at normal use condition or at accelerated condition only. Since the lifetimes of the test items follow the Rayleigh distribution, the probability density function of an item tested at normal use condition is given as above

$$f(t)=\frac{t}{\theta^2}\exp\left(-\frac{t^2}{2\theta^2}\right), \qquad 0\le t<\infty, \theta>0$$

ISSN: 2277-9655
Scientific Journal Impact Factor: 3.449
(ISRA), Impact Factor: 1.852

And for an item tested at accelerated use condition, the pdf is given by

$$f(x)=\frac{\beta^2 x}{\theta^2}\exp\left(-\frac{(\beta x)^2}{2\theta^2}\right), \qquad 0\le x<\infty, \theta>0$$

where $X=\beta^{-1}T$, β is the acceleration factor

As the type-II censoring test terminates when a predetermined number of failures r is reached, so, the observed lifetimes $t_{(1)}\le\ldots\ldots\le t_{(n_u)}\le\tau$ and $t_{(1)}\le\ldots\ldots\le t_{(n_a)}\le\tau$ are ordered failure times at normal use and accelerated conditions respectively, where τ is the time of the r^{th} failure at which the test is terminated, n_u and n_a are the numbers of items failed at normal use and accelerated use conditions, respectively which are given by

$$n_u=\sum_{i=1}^{n(1-k)}\delta_{ui} \text{ and } n_a=\sum_{j=1}^{nk}\delta_{aj}$$

Let δ_{ui} and δ_{aj} be the indicator functions such that

$$\delta_{ui}=\begin{cases}1 & t_i\le\tau \\ 0 & otherwise\end{cases} \qquad i=1,2,\ldots\ldots n(1-k)$$

And

$$\delta_{aj}=\begin{cases}1 & x_j\le\tau \\ 0 & otherwise\end{cases} \qquad j=1,2,\ldots\ldots nk$$

Then the likelihood function for (t_i,δ_{ui}), the likelihood function for (x_j,δ_{aj}) and the total likelihood function for $(t_1;\delta_{u1},\ldots\ldots t_{n(1-k)};\delta_{un(1-k)}, x_1;\delta_{a1},\ldots\ldots, x_{nk};\delta_{ank})$ are respectively given by

$$L_{ui}(t_i,\delta_{ui}\mid\theta)=\prod_{i=1}^{n(1-k)}\left\{\frac{t_i}{\theta^2}\exp\left(-\frac{t_i^2}{2\theta^2}\right)\right\}^{\delta_{ui}}\left\{\exp\left(-\frac{\tau^2}{2\theta^2}\right)\right\}^{\bar{\delta}_{ui}} \quad (4)$$

$$L_{aj}(x_j,\delta_{aj}\mid\beta,\theta)=\prod_{j=1}^{nk}\left\{\frac{\beta^2 x_j}{\theta^2}\exp\left(-\frac{(\beta x_j)^2}{2\theta^2}\right)\right\}^{\delta_{aj}}\left\{\exp\left(-\frac{(\beta\tau)^2}{2\theta^2}\right)\right\}^{\bar{\delta}_{aj}} \quad (5)$$

$$L(t,x,\mid\beta,\theta)=\prod_{i=1}^{n(1-k)}\left\{\frac{t_i}{\theta^2}\exp\left(-\frac{t_i^2}{2\theta^2}\right)\right\}^{\delta_{ui}}\left\{\exp\left(-\frac{\tau^2}{2\theta^2}\right)\right\}^{\bar{\delta}_{ui}}$$

$$\prod_{j=1}^{nk}\left\{\frac{\beta^2 x_j}{\theta^2}\exp\left(-\frac{(\beta x_j)^2}{2\theta^2}\right)\right\}^{\delta_{aj}}\left\{\exp\left(-\frac{(\beta\tau)^2}{2\theta^2}\right)\right\}^{\bar{\delta}_{aj}} \quad (6)$$

where $\bar{\delta}_{ui}=1-\delta_{ui}$ and $\bar{\delta}_{aj}=1-\delta_{aj}$

Taking log of above equation

$$l=\log L=\sum_{i=1}^{n(1-k)}\delta_{ui}\left[\log t_i-2\log\theta-\frac{t_i^2}{2\theta^2}\right]-\frac{\tau^2}{2\theta^2}\sum_{i=1}^{n(1-k)}(1-\delta_{ui})$$

$$\sum_{j=1}^{nk}\delta_{aj}\left[\log x_j+2\log\beta-2\log\theta-\frac{\beta^2x_j^2}{2\theta^2}\right]-\frac{\beta^2\tau^2}{2\theta^2}\sum_{j=1}^{nk}(1-\delta_{aj})$$

(7)

β and θ are obtained by solving the equations $\frac{\partial l}{\partial\beta}=0$ and $\frac{\partial l}{\partial\theta}=0$

$$\frac{\partial l}{\partial\theta}=-\frac{2r}{\theta}+\frac{1}{\theta^3}\sum_{i=1}^{n(1-k)}\delta_{ui}t_i^2+\frac{\tau^2}{\theta^3}\{n(1-k)-nu\}+\frac{\beta^2}{\theta^3}\sum_{j=1}^{nk}\delta_{aj}x_j^2 + \frac{\beta^2\tau^2}{\theta^3}(nk-n_a)=0 \quad (8)$$

$$\frac{\partial l}{\partial\beta}=\frac{2n_a}{\beta}-\frac{\beta}{\theta^2}\sum_{j=1}^{nk}\delta_{aj}x_j^2-\frac{\beta\tau^2}{\theta^2}(nk-n_a)=0 \quad (9)$$

From the above equations

$$\hat{\theta}=\left[\frac{s_1+\beta^2 s_2}{2r}\right]^{1/2} \quad (10)$$

$$\hat{\beta}=\theta\left[\frac{2n_a}{s_2}\right]^{1/2} \quad (11)$$

Where $s_1=\sum_{i=1}^{n(1-k)}\delta_{ui}t_i^2+\tau^2\{n(1-k)-nu\}$ and

$$s_2=\sum_{j=1}^{nk}\delta_{aj}x_j^2+\tau^2(nk-n_a)$$

It is difficult obtain a closed form solution to nonlinear equations. The Newton-Raphson method is

ISSN: 2277-9655
Scientific Journal Impact Factor: 3.449
(ISRA), Impact Factor: 1.852

used to obtain the ML estimate of β .Once the value of $\hat{\beta}$ is obtained from equation (11), the MLE of θ can easily be obtained by the equation (10).

Asymptotic confidence interval

The asymptotic variance-covariance matrix of the ML estimators of the parameters can be approximated by numerically inverting the Fisher-information matrix F. It is composed of the negative second derivatives of the natural logarithm of the likelihood function evaluated at the ML estimates. Therefore, the asymptotic Fisher-information matrix can be written as above

$$F=\begin{bmatrix}-\frac{\partial^2 l}{\partial\theta^2} & -\frac{\partial^2 l}{\partial\theta\partial\beta}\\ -\frac{\partial^2 l}{\partial\beta\partial\theta} & -\frac{\partial^2 l}{\partial\beta^2}\end{bmatrix}$$

Elements of Fisher Information matrix are

$$\frac{\partial^2 l}{\partial\theta^2}=\frac{2r}{\theta^2}-\frac{3}{\theta^4}\left[s_1+\beta^2 s_2\right]$$

$$\frac{\partial^2 l}{\partial\beta^2}=-\frac{2n_a}{\beta^2}-\frac{s_2}{\theta^2}$$

$$\frac{\partial^2 l}{\partial\theta\partial\beta}=\frac{\partial^2 l}{\partial\beta\partial\theta}=\frac{2\beta}{\theta^3}s_2$$

The variance covariance and covariance matrix of the parameter can be written as

$$\Sigma=\begin{bmatrix}-\frac{\partial^2 l}{\partial\theta^2} & -\frac{\partial^2 l}{\partial\theta\partial\beta}\\ -\frac{\partial^2 l}{\partial\beta\partial\theta} & -\frac{\partial^2 l}{\partial\beta^2}\end{bmatrix}^{-1}=\begin{bmatrix}AVar(\hat{\theta}) & ACov(\hat{\theta}\hat{\beta})\\ ACov(\hat{\beta}\hat{\theta}) & AVar(\hat{\beta})\end{bmatrix}$$

The $100(1-\xi)\%$ asymptotic confidence interval for θ and β are respectively given by

$$\left[\hat{\theta}\pm Z_{1-\frac{\xi}{2}}\sqrt{AVar(\hat{\theta})}\right] \text{ and } \left[\hat{\beta}\pm Z_{1-\frac{\xi}{2}}\sqrt{AVar(\hat{\beta})}\right]$$

Simulation study

In order to obtain the MLEs of θ and β and to study the properties of these estimates through Mean squared errors (MSEs), and variance of the estimators, a simulation study is performed. Furthermore, the asymptotic variance and covariance matrix and the two-sided confidence intervals of the model parameters are obtained. For this task following steps are involved which are as follows:

- Several data sets generated from Rayleigh distribution under type-II censored data are considered with sample sizes 100, 200, 300, 400 and 500 using 500 replications for each sample size. The combinations (θ,β) of values of the parameters are chosen to be (4, 1.6) and (5, 1.3).
- Under type II censoring, choose a proportion of sample units allocated to accelerated condition to be $k=40\%$ and the number of the failure units $r=.80n$ (the test will terminate after 80% of the test units failed), where the censoring time of failure constant stress PALT is τ .
- The Newton Rephson method is used to obtain the MLEs of θ and β . Measure of accuracy like Mean Squared Error (MSE) and variance are calculated for checking the performance of the estimator.
- Finally 95% confidence interval and probability coverage of the estimators are calculated.

ISSN: 2277-9655
Scientific Journal Impact Factor: 3.449
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Table 1
Simulations result based on censoring data in constant stress PALT with for the parameters (θ,β) set as (4, 1.6), given k=0.40 and r=0.8n

n	Parameters $\begin{pmatrix}\hat{\theta}\\ \hat{\beta}\end{pmatrix}$	MSE	Variance	95% LCL	95% UCL	95% confidence Interval coverage
100	4.9358	0.0145	3.9150	3.7532	4.6631	0.94607
	1.6258	0.0701	1.7595	1.2689	1.9817	0.93546
200	4.6551	0.0269	2.3591	3.7293	4.2375	0.95714
	1.6121	0.0397	0.9889	1.3431	1.9659	0.94875
300	4.2314	0.0201	1.5212	3.8101	4.1592	0.95434
	1.6105	0.0256	0.4658	1.3563	1.9393	0.95854
400	4.1915	0.0157	1.1491	3.8954	4.0795	0.95644
	1.6094	0.0184	0.2901	1.3683	1.8700	0.95463
500	4.0942	0.0198	1.0299	3.9216	4.0289	0.95539
	1.6071	0.0132	0.1874	1.4988	1.7421	0.95368

Table 2
Simulations result based on censoring data in constant stress PALT with for the parameters (θ,β) set as (5, 1.3), given k=0.40 and r=0.8n.

n	Parameters $\begin{pmatrix}\hat{\theta}\\ \hat{\beta}\end{pmatrix}$	MSE	Variance	95% LCL	95% UCL	95% confidence Interval coverage
100	6.1074	0.0294	6.9485	4.5763	5.5602	0.94286
	1.4561	0.0834	1.9039	1.1460	1.8841	0.94986
200	5.8533	0.0214	2.9071	4.6202	5.4185	0.95798
	1.3909	0.0627	0.9884	1.1849	1.7940	0.94804
300	5.2801	0.0185	1.6050	4.6955	5.2894	0.95649
	1.3472	0.0290	0.6795	1.2518	1.7539	0.95749
400	5.1192	0.0139	0.5639	4.8997	5.2170	0.95358
	1.3198	0.0155	0.1495	1.2853	1.5841	0.95777
500	5.1042	0.0122	0.2744	4.9002	5.1968	0.96683
	1.3059	0.0118	0.1038	1.2905	1.4507	0.96365

Conclusion

This study considered the problem of estimation of simple constant stress PALT for the Rayleigh distribution under type-II censored data. From the table (1) and (2), it is observed that that the ML estimates approximate the true values of the parameters as the sample size n increases. Also, we find that, for a fixed θ and β the mean squared errors and asymptotic variances of the estimators are decreasing as the sample size n is getting to be large. It is also noticed that when the sample size increases, the interval of the estimators are decreases and probabilities coverage are close to the nominal level and do not change much across the different sample sizes.

Thus, it is reasonable to say that the proposed model work well which enables to save time and money considerably without using a high stress to all test units.

ISSN: 2277-9655
Scientific Journal Impact Factor: 3.449
(ISRA), Impact Factor: 1.852

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